A Lie Algebraic Model Predictive Control for Legged Robot Control: Implementation and Stability Analysis

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Abstract— **Rigid body model has been widely adopted to plan the centroidal motion of legged robot. However, the rigid body motion evolves on the SE(3) manifold, which is nonlinear and not trivial to parameterize. The existing Model Predictive Control (MPC) either uses local parameterization that suffers from singularities, or geometric based liearization that makes the problem state dependent or hard to vectorize. In this work, we focus on the MPC on matrix Lie group. We linearize the configuration error and design a quadratic cost function in its Lie algebra. Given an initial condition, the linearized configuration error dynamics is globally valid and evolve independently of the system trajectory. The quadratic cost function could also ensure exponential convergence rate. The proposed MPC has been experimentally validated in MIT mini Cheetah locomotion and pose control.**

Paper Type **- Recent work [1], [2].**

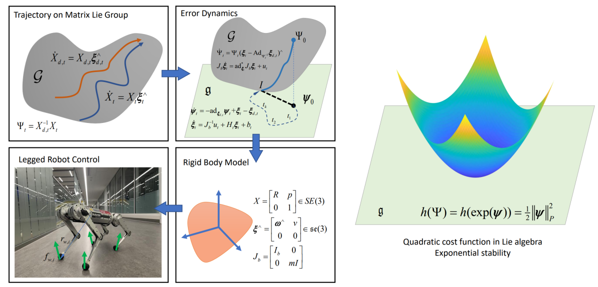

Fig. 1: The proposed error-state MPC framework. The tracking error is defined on a matrix Lie group and linearized in the Lie algebra. A convex MPC algorithm is derived via the linearized dynamics for tracking control. The proposed algorithm is applied to a single rigid body system and verified on a quadrupedal robot MIT Mini Cheetah. A quadratic cost function in Lie algebra can verify the exponential stability of the proposed MPC.

I. Introduction

The geometry of the configuration space of a robotics system can naturally be modeled using matrix Lie (continuous) groups [3], [4]. For example, the centroidal dynamics of legged robots can be approximated by a single rigid body, whose motion is on $\mathrm{SE}(3)$.

The Euler angle based convex Model Predictive Control (MPC) [5] has been proposed for locomotion planning on the quadrupedal robot. Zero roll and pitch angle assumptions are validated by assuming a flat ground, which may fail when such assumptions no longer hold. To avoid the problem, the geometric MPC that utilize the symmetry of the Lie group has been proposed. A local control law has been proposed in [6], [7], where the linearized dynamics are defined by a local diffeomorphism from the $\mathrm{SE}(3)$ manifold to $\mathbb{R}^n$ space. However, such a diffeomorphism is not unique and too abstract for controller design. The Variational Based Linearization (VBL) technique [8] are applied to linearize the Lagrangian to obtain the discrete-time equation of motion and applied to robot pose control [9]. A VBL based MPC is proposed in [10] for locomotion on discrete terrain using a gait library. The result suggests that the VBL based linearization can preserve the energy, thus making the system more stable. However, the VBL method linearized the system at the reference trajectory, which may result in unstable motion [11]. Other than linearizing at the reference trajectory, the work of [11] linearized the system at the current operating point to obtain the Quadratic Programming (QP) problem for tracking of legged robot trajectory. However, the linearized state matrix of [11] depends on the orientation, which can be avoided by exploiting the symmetry of the system as done in this work.

This paper summarizes our pre-prints [1] about the MPC and [2] about the stability analysis. In particular, the main contributions are as follows:

1) We derive the linearized configuration error dynamics and equations of motion in the Lie algebra (tangent space at the identity) that, given an initial condition, are globally valid and independent of the system trajectory.
2) We develop a convex MPC algorithm for the tracking control problem using the linearized error dynamics, which can be solved efficiently using QP solvers.
3) The proposed controller is validated in experiments on quadrupedal robot pose control and locomotion.
4) The exponential stability can be verified by a quadratic Lyapunov function expressed in the Lie algebra.

II. Math Preliminary

This section provides a brief overview of the necessary mathematical background used in the developed approach.

Let $\mathcal{G}$ be an n-dimensional matrix Lie group and $\mathfrak{g}$ its associate Lie algebra (hence, $\dim \mathfrak{g} = n$) [12], [13]. For convenience, we define the following isomorphism

$$(\cdot)^\wedge : \mathbb{R}^n \to \mathfrak{g}, \tag{1}$$

that maps an element in the vector space $\mathbb{R}^n$ to the tangent space of the matrix Lie group at the identity. Then, for any $\phi \in \mathbb{R}^n$, we can define the Lie exponential map as

$$\exp(\cdot) : \mathbb{R}^n \to \mathcal{G}, \quad \exp(\phi) = \exp_{\mathrm{m}}(\phi^\wedge), \tag{2}$$

where $\exp_{\mathrm{m}}(\cdot)$ is the exponential of square matrices. The Lie logarithmic map as the inverse of Lie exponential map

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is defined as:

$$\log(\cdot) : \mathcal{G} \to \mathbb{R}^n. \tag{3}$$

For every $X \in \mathcal{G}$, the adjoint action, $\mathrm{Ad}_X : \mathfrak{g} \to \mathfrak{g}$, is a Lie algebra isomorphism that enables change of frames

$$\mathrm{Ad}_X(\phi^\wedge) = X\phi^\wedge X^{-1}. \tag{4}$$

Its derivative at the identity gives rise to the adjoint map in the Lie Algebra as

$$\mathrm{ad}_\phi \eta = [\phi^\wedge, \eta^\wedge], \tag{5}$$

where $\phi^\wedge, \eta^\wedge \in \mathfrak{g}$ and $[\cdot,\cdot]$ is the Lie Bracket.

Consider the motion of an object whose state space is a Lie group $\mathcal{G}$. We define a left-invariant Lagrangian $\mathcal{L} : \mathfrak{g} \to \mathbb{R}$:

$$\mathcal{L}(\xi) = \frac{1}{2}\xi^\mathsf{T} J_b \xi,$$

where ξ is the twist in the body frame, and J_b is the generalized inertia matrix in the body fixed principal axes. Given the left invariant Lagrangian, We can then write the *forced* Euler-Poincaré equations [14] as

$$J_b\dot{\xi} = \mathrm{ad}^*_\xi J_b \xi + u, \tag{6}$$

where $u \in \mathfrak{g}^*$ is the generalized control input force applied to the body fixed principal axes, ad^* is the co-adjoint action, and $\mathfrak{g}^*$ is the cotangent space.

For tracking control on Lie group $\mathcal{G}$, we define the desired trajectory as $X_{d,t} \in \mathcal{G}$ and the actual state as $X_t \in \mathcal{G}$, both as function of time t. Given the twists ξ_t and desired twists $\xi_{d,t}$ and the reconstruction equation, we have

$$\frac{d}{dt}X_t = X_t\xi_t^\wedge, \ \frac{d}{dt}X_{d,t} = X_{d,t}\xi_{d,t}^\wedge.$$

Similar to the left or right error defined in [15], we define the error between X_t^d and X_t as

$$\Psi_t = X_{d,t}^{-1}X_t \in \mathcal{G}. \tag{7}$$

For the tracking problem, our goal is to drive the error from the initial condition Ψ_0 to the identity $I \in \mathcal{G}$. Taking derivative on both sides of (7), we have

$$\begin{aligned}\frac{d}{dt}\Psi_t = \dot{\Psi}_t &= \frac{d}{dt}(X_{d,t}^{-1})X_t + X_{d,t}^{-1}\frac{d}{dt}X_t \\ &= X_{d,t}^{-1}\frac{d}{dt}X_t - X_{d,t}^{-1}\frac{d}{dt}(X_{d,t})X_{d,t}^{-1}X_t \\ &= X_{d,t}^{-1}X_t\xi_t^\wedge - X_{d,t}^{-1}X_{d,t}\xi_{d,t}^\wedge X_{d,t}^{-1}X_t = \Psi_t\xi_t^\wedge - \xi_{d,t}^\wedge\Psi_t.\end{aligned}$$

Therefore,

$$\dot{\Psi}_t = \Psi_t(\xi_t - \Psi_t^{-1}\xi_{d,t}\Psi_t)^\wedge = \Psi_t(\xi_t - \mathrm{Ad}_{\Psi_t^{-1}}\xi_{d,t})^\wedge, \tag{8}$$

III. Error-State Convex MPC

A. System linearization

Recall that we can map the error from the Lie Algebra to the group element by the group exponential map. We define $\psi_t^\wedge$ as an element of the Lie Algebra that corresponds to Ψ_t. Thus by the exponential map, we have

$$\Psi_t = \exp(\psi_t), \ \Psi_t \in \mathcal{G}, \ \psi_t^\wedge \in \mathfrak{g}.$$

Given the first-order approximation of the exponential map,

$$\Psi_t = \exp(\psi_t) \approx I + \psi_t^\wedge,$$

and a first-order approximation of the adjoint map

$$\mathrm{Ad}_{\Psi_t} \approx \mathrm{Ad}_{I+\psi_t{}^\wedge},$$

we can linearize (8) by dropping the second-order terms as

$$\dot{\Psi}_t \approx (I + \dot{\psi}_t^\wedge) \approx (I+\psi_t^\wedge)(\xi_t - \mathrm{Ad}_{(I-\psi_t^\wedge)}\xi_{d,t})^\wedge, \tag{9}$$

$$\dot{\psi}_t = -\mathrm{ad}_{\xi_{d,t}}\psi_t + \xi_t - \xi_{d,t}. \tag{10}$$

Equation (10) is the linearized velocity error in the Lie algebra.

Remark 1. *Lifting the problem to the Lie algebra vectorizes the dynamics without complicated manipulations.*

The dynamics of ξ_t is described by (6), which is nonlinear. To compute a locally linear approximation of the nonlinear term, we adopt the Jacobian linearization around the operating point $\bar{\xi}$

$$J_b\dot{\xi} \approx \mathrm{ad}^*_{\bar{\xi}} J_b\bar{\xi} + \frac{\partial \mathrm{ad}^*_\xi J_b \xi}{\partial \xi}|_{\bar{\xi}}(\xi - \bar{\xi}) + u. \tag{11}$$

Thus, we have the linearized dynamics in the following form

$$\dot{\xi} = H_t\xi + J_b^{-1}u + b_t, \tag{12}$$

We define the system states as $x_t := \begin{bmatrix}\psi_t \\ \xi_t\end{bmatrix}$. Then, the linearized dynamics becomes

$$\dot{x}_t = A_t x_t + B_t u_t + h_t, \tag{13}$$

where

$$A_t := \begin{bmatrix} -\mathrm{ad}_{\xi_{d,t}} & I \\ 0 & H_t \end{bmatrix}, B_t := \begin{bmatrix} 0 \\ J_b^{-1}\end{bmatrix}, h_t := \begin{bmatrix} -\xi_{d,t}, \\ b_t \end{bmatrix}.$$

Remark 2. *The operating point $\bar{\xi}$, for computing H and b, need not to be the reference trajectory $\xi_{d,t}$. In following sections, we set the operating point at the current system states when the controller is applied, which exhibit higher stability as shown by [11].*

Remark 3. *Up to now, the linearization is general for any Lie group system. For the implementation in SE(3) rigid body, the specific form of the matrices can be found in [1].*

B. Convex MPC design

On Lie groups, our cost function is designed to regulate the tracking error ψ_t and its derivative $\dot{\psi}_t$ rather than the difference between $\xi_{d,t}$ and ξ_t. Thus, our tracking error can be designed as:

$$y_t := \begin{bmatrix}\psi_t \\ \dot{\psi}_t\end{bmatrix} = \begin{bmatrix} I & 0 \\ -\mathrm{ad}_{\xi_{d,t}} & I\end{bmatrix} x_t - \begin{bmatrix} 0 \\ \xi_{d,t}\end{bmatrix}, \tag{14}$$

Given some semi-positive definite weights P, Q, and R, we can now write the quadratic cost function as

$$N(y_{t_f}) = y_{t_f}^\mathsf{T} P y_{t_f}, \ L(y_t, u_t) = y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t. \tag{15}$$

Given the future twists $\xi_{d,t}$, initial error state ψ_0 and twist ξ_0, we can define all the matrices. Discretizing the system at time steps $\{t_k\}_{k=1}^{N}$, we can design the MPC as follows.

Problem 1. *Find $u_k \in \mathfrak{g}^*$ such that*

$$\begin{aligned}
\min_{u_k} \quad & y_N^\mathsf{T} P y_N + \sum_{k=1}^{N-1} y_k^\mathsf{T} Q y_k + u_k^\mathsf{T} R u_k \\
\textit{s.t.} \; & x_{k+1} = A_k x_k + B_k u_k + h_k \\
& u_k \in \mathcal{U}_k, x_0 = x(0) \\
& k = 0, 1, \ldots, N-1.
\end{aligned}$$

In Problem 1, A_k, B_k, and h_k can be obtained by zero-order hold or Euler first-order integration. Problem 1 is a QP problem that can be solved efficiently, e.g., using OSQP [16].

IV. Stability analysis

The stability of the proposed controller could be verified by a quadratic Lyapunov cost function in Lie algebra. First, we introduce the left invariant inner product that defines the inner product in different tangent space.

Definition 1. *Given $\phi_1, \phi_2 \in \mathbb{R}^{\dim \mathfrak{g}}$ and $\phi_1^\wedge, \phi_2^\wedge \in \mathfrak{g}$, we define the inner product $\langle \phi_1^\wedge, \phi_2^\wedge \rangle_\mathfrak{g} = \phi_1^\mathsf{T} P \phi_2$, where P is a positive definite matrix. This inner product is left-invariant. To see this, suppose $X\phi_1^\wedge, X\phi_2^\wedge \in T_X\mathcal{G}$, $\forall X \in \mathcal{G}$, then*

$$\begin{aligned}
\langle X\phi_1^\wedge, X\phi_2^\wedge \rangle_X &= \langle (\ell_{X^{-1}})_* X\phi_1^\wedge, (\ell_{X^{-1}})_* X\phi_2^\wedge \rangle_\mathfrak{g} \\
&= \langle \phi_1^\wedge, \phi_2^\wedge \rangle_\mathfrak{g},
\end{aligned}$$

where $(\ell_{X^{-1}})_* = X^{-1} : T_X\mathcal{G} \to \mathfrak{g}$ is the pushforward map.

Then, we could derive the gradients of the quadratic cost function in the tangent space.

Theorem 1. *Consider the state $X \in \mathcal{G}$, $\phi \in \mathbb{R}^{\dim \mathfrak{g}}$, and $X = \exp(\phi)$. We consider the metric in Definition 1. The function $h = \frac{1}{2}\|\phi\|_P^2$ is a candidate Lyapunov function and the gradient of h with respect to X is*

$$\nabla h = X\phi^\wedge. \tag{16}$$

Finally, we show that a linear feedback in Lie algebra could regulate the state to the identity exponentially.

Theorem 2. *Consider the state in Theorem 1 as a trajectory. Let $\xi^\wedge \in \mathfrak{g}$. The system*

$$\dot{X} = X\xi^\wedge$$

can be exponentially stabilized to $X = I$ by linear feedback $\xi = -K\phi$, where K is a gain matrix with only positive eigenvalues.

The detailed proof of the theorems are presented in [2]. For the proposed MPC we could follow the same steps and estimate the region of attraction. For the unconstrained case, the resulting LQR problem will lead to a linear feedback, whose stability property can be verified by Theorem. 2.

V. Validation on quadrupedal robot

We now conduct two experiments on the quadrupedal robot Mini Cheetah [17] to evaluate the proposed MPC. Both experiments use a single rigid body model to approximate the torso motion. We apply MIT controller [5] with the proposed MPC to plan the ground reaction force (GRF).

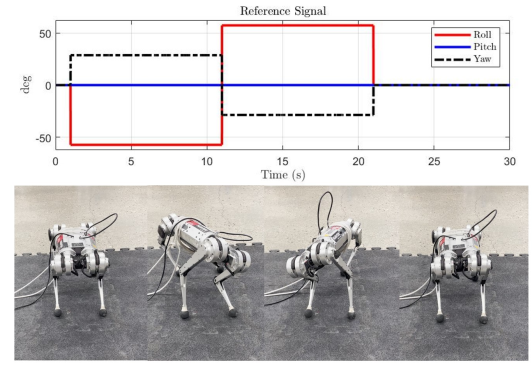

Fig. 2: Reference signal for roll and yaw angle tracking. From 1 to 11 secs, the robot roll changes from 0 to -57.3 degree and yaw changes from 0 to 28.5 degree. Then the robot leans to the opposite side for 10 seconds.

A. Robot pose tracking

In this experiment, a mixture of roll and yaw reference angle is applied for tracking. The reference signals and snapshots of robot motion are presented in Fig. 2. Each controller is implemented three times. The details of the responses are presented in Fig. 3. It can be seen that as no feedforward force at the equilibrium is provided, all controllers have steady-state error. However, the geometric-based controller, i.e., proposed and the VBL based MPC, has a smaller steady-state error than the Euler angle-based one. As the VBL based MPC does not conserve the scale of the error, the convergence rate is much lower than our controller, especially when the opposite Euler angle signal is applied at the middle of the reference profile. As can been seen in the Fig. 3, at 11.5 s, the tracking performance of the VBL based method has an trakcing error that is 0.3 rad more than the proposed one.

B. Robot trotting

We also apply our controller to robot locomotion. Ours and baseline controllers are deployed to plan the robot's GRF given command twists. Then the GRF is applied to the Whole Body Impulse Control (WBIC) [18] to obtain the joint torques. Unlike the conventional whole-body controller, WBIC prioritizes the GRF generation by penalizing the deviation of GRF from the planned GRF. We increase the penalty for the GRF by 1e4 times in the original WBIC, so the GRF merely deviates from the planned one.

We first apply a step signal in yaw rate. Then we add a step signal in x motion in the robot frame, and the yaw rate becomes a sinusoidal signal. The reference is presented in Fig. 5 and the snapshots of the experiments are in Fig. 4. We find that ours and the VBL-MPC can better track the yaw rate than the Euler angles-based MPC, as expected. As the orientation and position tracking errors are small because every step is integrated from the current state, it is reasonable that all controllers perform well in position tracking. The result can be seen in Fig. 5.

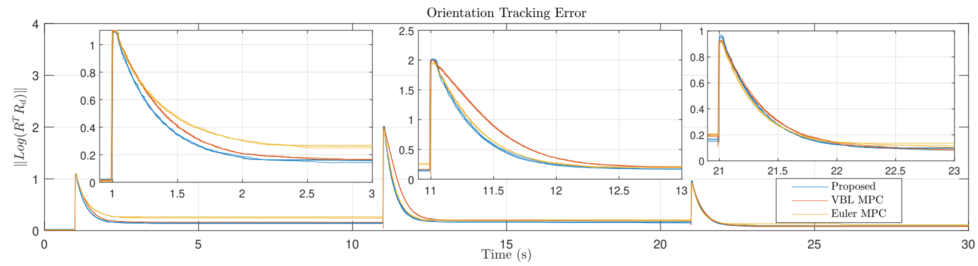

Fig. 3: Error convergence for roll and yaw tracking. When a new step signal is applied, our controller converges faster than the baseline methods and has a smaller steady-state error. The Euler angle-based MPC has a larger steady-state error as both roll and yaw signals are applied.

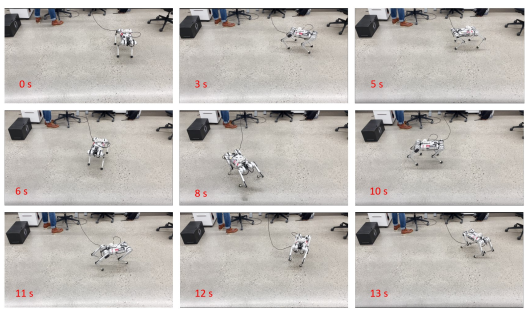

Fig. 4: Snapshots of the experiments on reference tracking in Mini Cheetah trotting. The time corresponds to the reference signal in Fig. 5.

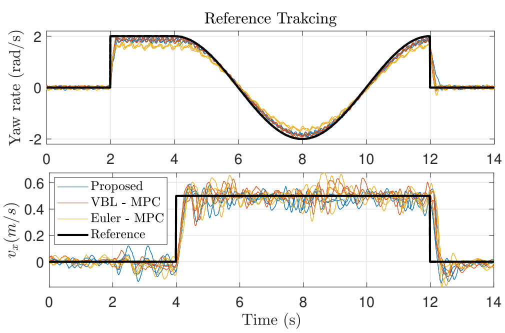

Fig. 5: Reference tracking for quadrupedal robot trotting. Each controller is tested three times. The responses are too noisy; thus, the results are smoothed using the moving average filter.

VI. Conclusions

We developed a new error-state Model Predictive Control approach on connected matrix Lie groups for robot control. By exploiting the existing symmetry of pose control problem on Lie group, we showed that the linearized tracking error dynamics and equations of motion in the Lie algebra are globally valid and evolve independently of the system trajectory. In addition, we formulated a convex MPC program for solving the problem efficiently using QP solvers. An Lyapunov function expressed in Lie algebra is introduced to verify the exponential stability of the proposed controller. The experimental results confirm that the proposed approach provides faster convergence when rotation and position are controlled simultaneously. Future work will implement the trajectory optimization using this geometric control framework proposed in [2] for legged robot control.

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